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A Convex Envelope Formula for Multilinear Functions*

ANATOLIY D. RIKUN
East Coast Product Group, 45 Winnett St., Hamden, CT 06517, U.S.A. (email: rikun@ptc.com)

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Abstract. Convex envelopes of multilinear functions on a unit hypercube are polyhedral. This well-known fact makes the convex envelope approximation very useful in the linearization of non-linear 0–1 programming problems and in global bilinear optimization. This paper presents necessary and sufficient conditions for a convex envelope to be a polyhedral function and illustrates how these conditions may be used in constructing of convex envelopes. The main result of the paper is a simple analytical formula, which defines some faces of the convex envelope of a multilinear function. This formula proves to be a generalization of the well known convex envelope formula for multilinear monomial functions.

Key words: Nonlinear 0–1 optimization, linearization, convex envelope, concave extension, bilinear programming, global optimization.

Introduction

A great deal of effort has been devoted to the field of finding tight convex piecewise linear approximation of multilinear functions, and in applying these approximations within the context of nonlinear programming [1–10, 14, 16, 17]. Most of the efforts were devoted to the case of multilinear functions (1) over the hypercube $U^n = [0, 1]^n$:

$$f(x) = \sum_{j \in N} \alpha_j \prod_{i \in I_j} x_i, \tag{1}$$

where $Ij \subset 1, \ldots, n; N = N_+ \cup N_- : \alpha_j > 0 \forall j \in N_+$ and $\alpha_j < 0 \forall j \in N_-$ and $\prod$ means the product. It is a well known fact [2], that any 0–1 problem may be rewritten in multilinear form, and, thus any opportunity to construct tighter linear underestimates** of multilinear functions may lead to more effective algorithms in global optimization of 0–1 problems. In addition, linear underestimating functions are of primary importance in global nonconvex bilinear programming [1,10,14,17].

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** Function $\Psi(x)$ is said to be a linear (or polyhedral) underestimate of function $f(x)$ on set *P* if $f(x) \geq \Psi(x) \forall x \in P$ and $\Psi(x)$ is made of a finite number of affine functions $hj(x) : \Psi(x) = \max\{h_j(x) | j = 1, ...J\}$.

The most common method of linearization of multilinear functions is based on the convex envelope* formula for the simplest monomial functions $\varphi(x) = x_1^* \cdots ^* x_n$ and $-\varphi(x)$ on U^n:

$$\mathit{conv}\varphi(x) = \max\left\{0, \sum x_i - n + 1\right\}\,, \tag{2}$$

$$\mathit{conv}\{-\varphi(x)\} = \max\{-x_i | i = 1, \ldots, n\}\,. \tag{3}$$

Thus, in ‘standard linearization’ $f^S(x)$ (as it was called in [4,9]) each monomial term in (1) is changed by the corresponding envelope (2) or (3):

$$f^s(x) = \sum_{j \in N_+} a_j \max\left\{0, \sum_{i \in N_+} x_j - |I_j - 1|\right\} + \sum_{j \in N_-} a_j \max\{-x_j | j \in I_j\} \tag{4}$$

Unfortunately, the standard approach often leads to very poor approximation and, in addition, it needs many more hyperplanes than convex envelope function does. To illustrate this fact, let us consider the following simple function of n variables:

$$\varphi(x) = \left(\sum_{i \neq j} x_i x_j\right) \Big/ 2\,. \tag{5}$$

As it will be shown further, the convex envelope of this function on U^n is:

$$\mathit{conv}\varphi(x) = \max\left\{0,_{k-1} C_1 \sum_1^n x_i -_k C_2 | k = 2, \ldots, n\right\}\,. \tag{6}$$

The standard approach leads to the approximation:

$$\varphi^s(x) = 0.5 \sum_{i \neq j} \max\{0, x_i + x_j - 1\}\,. \tag{7}$$

The precise error bounds for these two approaches are

$$\max\{\varphi(x) - \varphi^s(x) | x \in U^n \leq (n^2 - n)/8\,,$$

$$\max\{\varphi(x) - \mathit{conv}\varphi(x) | x \in U^n \leq (n-1)/8\,.$$

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* Convex envelope $\mathit{conv}_P \Psi(x)$ of a function $\Psi(x)$ on a closed convex set *P* is, by definition, the tightest convex underestimating function of $\Psi(x)$ on *P* [15]. Convex envelope is said to be a *polyhedral function* [15] if it is made of a finite set of lower bounding affine functions $h_j(x)$: $\mathit{conv}_P \Psi(x) = \max\{h_j(x) | j = 1, \ldots J\}$. We will refer to each functions $h_i(x)$ from which the convex envelope is made of as to *elements of convex envelope*: i.e. affine function $h(x)$ is an element of $\mathit{conv}_P \Psi(x)$ if there exists an open set $O \subset \mathit{dom}\Psi$ such that $h(x) = \mathit{conv}_P \Psi(x) \forall x \in O$.

Thus, the standard approach leads to poorer approximation and, at the same time, it needs many more hyperplanes for its realization as compared to convex envelope*.

In the case of bilinear programming problems, there exists another very important reason for using convex envelopes in branch and bound type methods. Let *P* be a hyperparallelepiped and *conv*$_p\Psi(x)$ has been found. Let in the branch-and-bound algorithm the set *P* be divided into subsets *Q* and *R* with inequalities $x_i \geq \alpha_i$ and $\alpha_i \geq x_i$ respectively. In this case the knowledge of conv$_p\Psi(x)$ allows easily to obtain the functions conv$_Q\Psi(x)$ and conv$_R\Psi(x)$, because all the elements of both conv$_Q\Psi(x)$ and conv$_R\Psi(x)$ may be obtained from the elements of conv$_P\Psi(x)$ by linear transformation. Thus, the difficult problem of calculating the convex envelope needs to be solved only once.

Unfortunately, finding the convex envelope of a multilinear function on a unit hypercube is a NP-hard problem [5] and convex envelope itself may be made of an extremely large number of elements**. Nevertheless, as the example above illustrates, the use of convex envelopes may give significant benefits both in accuracy and in the number of linear functions.

In this paper we will consider the following topics:

1. The necessary and sufficient conditions for the polyhedrality of convex envelopes. These conditions may be helpful to check if a polyhedral function coincides with the convex envelope. In particular, I. Crama in his recent paper [4] analyzed situations when the standard approach leads to the convex envelopes for multilinear 0–1 functions on U^n. Our results generalize his main result for the case of arbitrary multilinear functions on arbitrary convex polytope.
2. An analytical formula, which sometimes help to obtain elements of convex envelope of general multilinear functions. In particular, (2) and (3) may be easily obtained from this formula.

1. Necessary and Sufficient Conditions of Polyhedrality of Convex Envelope

Let $f(x)$ be a lower semicontinuous function on a compact polytope $P \subset R^n$. For the sake of convenience, in this section we will assume that $f(x) < \infty \, \forall x \in P$, and $f(x) = \infty \, \forall x \notin P$.

It may be helpful to use conv$_P f(x)$ to linearize some nonlinear function $f(x)$ only in the case, when conv$_P f(x)$ is a polyhedral function, i.e. when conv$_P f(x)$ is made of a finite set of affine functions $h_j(x)$: conv$_P f(x) = \max\{h_j(x) | j \in J\}$.

Being a polyhedral function, conv$_P f(x)$ may be defined by its values at a finite set of points, $x \in P$. On the other hand, by definition, $h_j(x) \leq f(x) \, \forall x \in P$ and it is always possible to consider only such functions h_j which coincide with $f(x)$ in

* Note, that the first-level of recently introduced reformulation-linearization approach [17] also leads to approximation φ^s for function (5), while higher level approximations from this methods can lead to tighter envelopes.

** In fact, even much more simple problems of minimization of a bilinear function with one negative eigenvalue prove to be NP-hard [13].

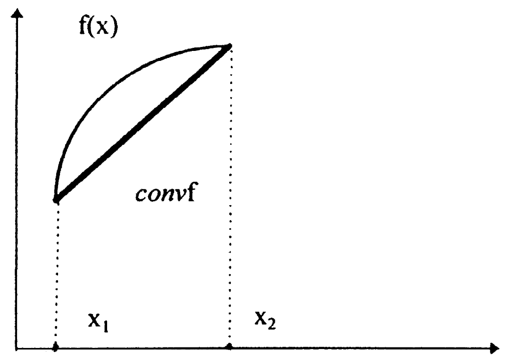

Figure 1.1. Graph of a function with a polyhedral convex envelope, $X(f) = \mathit{vertP} = x_1 \cup x_2$.

at least $n + 1$ affinely independent points of P. Thus, the question arises: how to determine *the minimal* (by inclusion) finite subset $X(f) \subset P$ so that $\mathrm{conv}_P f(x)$ would be completely defined by its values $f(x)$, at $x \in X(f)$?

It follows from the Caratheodory's theorem, that $f(x) = \mathit{conv}_P f(x)$ at each vertex x of polytope P (from now on we denote the set of all vertices as $vertP$). Thus, it is natural to analyze such sets $X(f)$ which contain $vertP$. The main result of this section is that for smooth functions which have polyhedral convex envelopes, $vertP = X(f)$, and, hence $conv_P f(x)$ of a smooth function $f(x)$ on a polytope P is completely defined by the values of this function at the extreme points of this polytope.

Now let us introduce some definitions; as usual, if φ is a real-valued function $\varphi : x \to R^n$, $epi(\varphi)$ denotes its epigraph:

$$\mathit{epi}(\varphi) = \{(z, x) | z \geq \varphi(x), x \in \mathrm{dom}(\varphi)\}, \text{ where } \mathit{dom}(\varphi) = \{x | \varphi(x) < \infty\}.$$

DEFINITION 1.1. Let $f(x)$ be a real valued lower semicontinuous function, defined on a convex set P, $\mathit{dom}(f) = P$. Set $X(f)$ is said to be a *generating set* of this function, if

$$X(f) = \{x | (x, f(x)) \in \mathit{vert}(\mathit{epi}(\mathit{conv}_P f(x)))\} \tag{8}$$

Thus, the generating set of a function $f(x)$ is the set of all x-coordinates of all vertices of the epigraph of the convex envelope of this function.

To illustrate the difference in generating sets of smooth functions which have polyhedral and non-polyhedral convex envelopes, let us consider a picture for the 1-dimensional case (both the convex envelope functions and the generating sets are typed in bold). As one can see from Figure 1.1, the generating set of the function, which has a polyhedral convex envelope coincides with the $\mathit{vertP} = x_1 \cup x_2$. In contrast with this, the nonpolyhedral convex envelope of smooth function has smooth nonlinear parts and its generating set contains the nonempty open subset

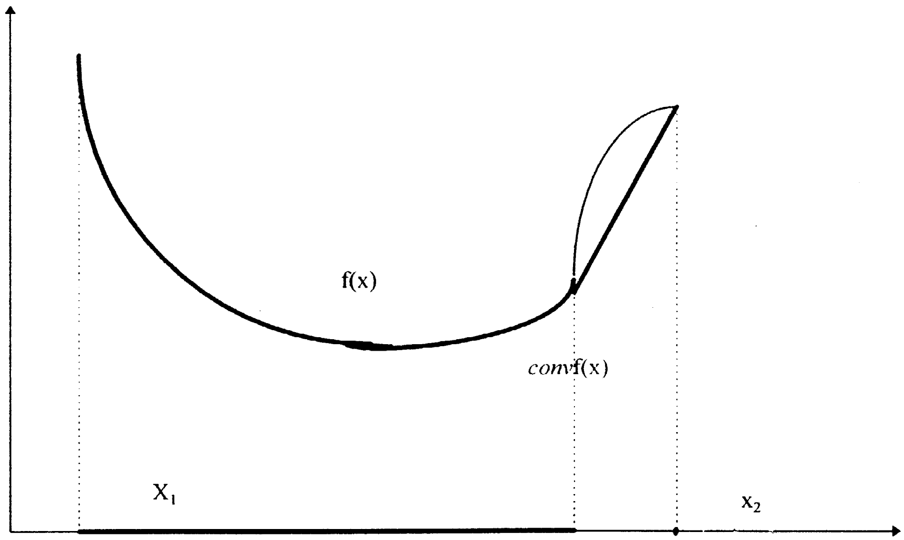

Figure 1.2. Graph of a function with nonpolyhedral convex envelope, $X(f) = X_1 \cup x_2$; *int*$X_1 \neq \emptyset$.

X_1 (see Figure 1.2). The same is true in the general case as it follows from Theorem 1.1.

DEFINITION 1.2. Function $f : P \to R^n$ is said to be continuously differentiable on convex polytope P if:
1. $f(x)$ is a continuous function on P.
2. for each $x \in P \backslash$ *vert*P and each F_x – the highest dimensional face of P containing x, function $f(x)$ is continuously differentiable on riF_x (where symbol riQ denotes the relative interior of set Q [15]).

THEOREM 1.1. *Let function $f(x)$ be continuously differentiable on a convex compact polytope P. The convex envelope of this function on P is a polyhedral function if and only if its generating set $X(f)$ coincides with the set of vertices of P:*

$$X(f) = vertP \tag{9}$$

REMARK 1.1. Because of *conv*$_P f(x) = f(x)\, \forall x \in$ *vert*P it is possible to give another equivalent formulation of the theorem. Let us define function f^∞ such that $f^\infty(x) = f(x)$ if $x \in$ *vert*P and $f^\infty(x) = \infty$ otherwise. Thus, the necessary and sufficient condition of the polyhedrality of the function *conv*$_P f(x)$ (9) may be rewritten in the form:

$$conv_P f(x) = conv_P\, f^\infty(x) \quad \forall x \in P \tag{10}$$

where *conv*$_P\, f^{\infty}(x)$ denotes the convex envelopes of the function $f^{\infty}(x)$. The equivalence of (9) and (10) follows from the self-evident fact that the generating set of f^{∞} coincides with the *vertP* $:\ X(f^{\infty}) =$ *vertP*.

Proof. The sufficiency is evident from the finiteness of the set of vertices of the polytope P.

To prove the necessity, let us assume the opposite. In this case there exists $x^0 \in X(f)\backslash vertP$. Let us consider first the case when x^0 is an internal point of P: $x^0 \in$ *intP*. Since x^0 belongs to the generating set but is not a vertex, there exists $m > 1$ different elements of *conv*$_P f(x)$, $H_1, \dots, H_m$ such that $H_i(x) = \langle h_i, x - x^0\rangle + f(x^0)\, \forall i = 1, \dots, m$; ($\langle v, u\rangle$ denotes scalar product of the vectors v, u) By definition of convex envelope, $H_i(x) \leq f(x)\, \forall x \in P$, $\forall i$. Thus, each H_i $i = 1, \dots, m$ is a support hyperplane and h_i is a subgradient of the function $f(x)$ at x^0. Because $f(x)$ is a differentiable function at x^0 its subgradients must coincide with its gradient [12]: i.e. $h_1 = \cdots = h_m = \partial f(x_0)/\partial x$. This equality contradicts to the fact that all $H_i(x)$ are different.

To complete the proof let us show that the same considerations are true in the case when $x^0 \notin$ *intP*. In this case let us consider F – the face of minimal dimension containing $x^0 : x^0 \in r$iF. Let us consider also the function f_F defined on *aff*(riF) – the affine hull of $riF\ :\ f_F(x) = f(x)\, \forall x \in$ *aff*(riF). All the conditions of the theorem are fulfilled for f_F instead of f, and for the polytope F instead of P and, besides, *conv*$_P f(x) =$ *conv*$_F f_F(x)$, $\forall x \in F^{\star}$. Using this we can apply the speculation mentioned above to the polytope F and the function f_F defined in the space *aff*(riF), and to the point x^0, which is an interior point of F in the *aff*(riF). Thus, we obtain the contradiction in the case, when $x^0 \notin$ *intP* as well. This contradiction completes the proof of the theorem. □

One aspect which makes finding the convex envelope hard is that the convex envelope of the sum of functions is not always equal to the sum of the convex envelopes of the addends (compare *conv*$\varphi(x)$ and $\varphi^s(x)$ for function (5)). Theorem 1.1 gives the criterion when the convex envelopes of the addends is equal to the polyhedral convex envelope of their sum.

COROLLARY 1.1. *Let functions $f_i(x)$ be continuously differentiable on convex compact polytope P, dom$(f_i) = P\, \forall i = 1, \dots, m$ and $f_0(x) = \sum_1^m f_i(x)$. Let the convex envelopes of all of these m functions $\{f_i(x)\}_{i=1}^m$ and their sum $f_0(x)$ be polyhedral. Then*

$$conv_P \left(\sum_1^m f_i(x) \right) = \sum_1^m conv_P f_i(x) \tag{11}$$

* This equality follows from the Caratheodory's theorem.

if and only if the generating set of $\sum_1^m \mathit{conv}_P f_i(x)$ *coincides with the set of vertices of P:*

$$X\left(\sum_1^m \mathit{conv}_P f_i(x)\right) = \mathit{vert}P\,. \tag{12}$$

REMARK 1.2. Corollary 1 gives a generalization of Crama's criterion of the equivalence between the standard and the convex envelope approaches in linearization of 0–1 multilinear function [4].

To use Corollary 1.1 one should check if the convex envelope of sum of the functions is a polyhedral one. The following statement gives a sufficient condition of polyhedrality of the convex envelope.

THEOREM 1.2. *Let* $f(x)$ *be a lower semicontinuous function on a compact polytope P and for every point* x_0 *which is not a vertex there exists a line* l_x *such, that* $f(x)$ *is a concave function in a neighborhood of* x_0 *on a segment* $(l_x \cap P)$ *and* $x_0 \in ri[l_x \cap P]$. *Then* $\mathit{conv}_P f(x)$ *is a polyhedral function and the equality (10) is true.*

Proof. Let us analyze the generating set $X(f)$ of this function. Let there exist $x_0 \in X(f)\backslash \mathit{vertP}$. In this case it is possible to find two other points x_1, x_2 : $x_1, x_2 \in [l_x \cap P]$ and $f(x_0) \geq (f(x_1) + f(x_2))/2$, $x_0 = (x_1 + x_2)/2$ because $f(x)$ is concave on $[x_1, x_2]$. Thus, $(x_0, f(x_0)) \notin \mathit{vert}(\mathit{epi}(\mathit{conv}_P f))$ and, as a result $x_0 \notin X(f)$. □

REMARK 1.3. Let $L : P \to R^1$ be a general multilinear function defined on the Cartesian product of the polytopes, $P = P_1, \ldots, P_k$, $P_i \in R^{n_i}$ i.e. function $L(x_1^0, \ldots, x_i, \ldots x_k^0)$ is linear function of n_i-dimensional vector x_i if all the other $k-1$ vector arguments $x_1^0, \ldots, x_k^0$ are fixed. It follows from the Theorem 1.2 that $\mathit{conv}_P L(x)$ is a polyhedral function. To prove this fact one can consider any line $l_x = \{x | x_j = x_j^0, j \neq i : x_i \notin \mathit{vert}(P_i)$ and $x_i = x_i^0 + t\xi\}$, where $t \in R^1$, $\xi \in F_x \subset P_i$, and F_x being facet of P_i which contains x_i^0.

REMARK 1.4. It is interesting to note that Theorem 1.2 may be reversed. Namely, let $f(x)$ be twice continuously differentiable on *P*. It is easy to prove that if $x \in X(f)$ and F_x be the biggest facet of *P* containing $x : x \in F_x$, then the Hessian $||\partial^2 f(x)/\partial x^2||$ must be nonnegatively definite on F_x.

This simple fact may be very useful in constructing convex envelopes. In particular, some formulas for the non-polyhedral convex envelopes presented in [16] could be obtained using this idea. As another example, let us find convex envelope of $f(x) = x_1^\alpha x_2 \ldots x_n$, $P = U^n$. If $\alpha > 1$, the function $f(x)$ is positively definite on the one face of $P : x_2 = \cdots = x_n = 1$ and thus, $X(f) = \{x | 0 < x_1 < 1; x_j = 1, j = 2, \ldots, n\} \cup \mathit{vert}U^n$. The application of Caratheodory's theorem to

$X(f)$ leads to the following formula [14]:

$$\mathit{conv}_P f(x) = \left(\sum_{i=1}^{n} x_i - n + 1\right)^{\alpha} \Bigg/ \left(\sum_{i=2}^{n} x_i - n + 2\right)^{\alpha-1}$$
$$\text{if } \Sigma x_i - n + 1 \geq 0 \text{ and } \mathit{conv}_P f(x) = 0 \text{ otherwise.}$$

This formula coincides with (2) if $\alpha = 1$.

REMARK 1.5. It should be noted that the function $f(x)$ in the theorem need not be continuous as well and Theorem 1.2 may be applied, for example, in the case of fixed charge problem.

In the future we will need a criterion to check if a specific affine function $h_i(x)$ is an element of $\mathit{conv}_P f(x)$.

LEMMA 1.1. *Let $f(x)$ be a continuously differentiable function on n-dimensional convex polytope $P \subset P^n$, and $\mathit{conv}_P f(x)$ be a polyhedral function. Let $h(x)$ be an affine function and there exist $n+1$ linearly independent vertices of $P : \xi_i, i = 1, \ldots n+1$ such that*

$$h(\xi_i) = f(\xi_i), \quad i = 1, \ldots, n+1 \tag{13}$$

and

$$h(x) \leq f(x), \quad \forall x \in \mathit{vertP} \tag{14}$$

then $h(x)$ is an element of the $\mathit{conv}_P f(x)$ and $h(x) = \mathit{conv}_S f(x) = \mathit{conv}_P f(x)$ $\forall x \in S$, where $S = \mathit{conv}\{\xi_i | i = 1, \ldots, n+1\}$.

Proof. Lemma 1.1 follows immediately from Caratheodory's theorem and equality (10). More specifically, let $x^0 \in S$. From Caratheodory's theorem we have:

$$\mathit{conv}_P f(x^0) \;=\; \min\left\{\sum_{i=1}^{n+1} \alpha_i f(x^i) \middle| x^0 = \sum_{i=1}^{n+1} \alpha_i x^i, \right. \\ \left. \sum_{i=1}^{n+1} \alpha_i = 1, x^i \in P, \alpha_i \geq 0, \quad \forall i = 1, \ldots, n+1 \right\}. \tag{15}$$

It follows from (10) that the minimum is achieved at some vertices of P: there exist $n+1$ points $\eta^j \in \mathit{vertP}$ and nonnegative α^i so that

$$\mathit{conv}_P f(x^0) = \sum_{i=1}^{n+1} \alpha_i f(\eta^i).$$

Because $h(x)$ is an affine function,

$$h(x^0) = \sum_{i=1}^{n+1} \alpha_i h(\eta^i)$$

and it follows from (14) that $h(x^0) \leq conv_P f(x^0)\,\forall x^0 \in P$. On the other hand, because of $x^0 \in S$, there exists $\alpha_i \geq 0 : \sum \alpha_i = 1, x^0 = \sum \alpha_i \xi_i$. Thus, $h(x^0) = h(\sum \alpha_i \xi_i) = \sum \alpha_i h(\xi_i) = \sum \alpha_i f(\xi_i) \geq conv_P f(x^0)\,\forall x^0 \in S$, and thus, $h(x^0) = conv_P f(x^0)$. □

Lemma 1.1 provides a way to check if a specific affine function is an element of the convex envelope. For example, it is easy to check that each function $h_k(x) = {}_{k-1}C_1 \sum_1^n x_i - {}_kC_2$ from (6) satisfies all conditions of the lemma and, thus it is an element of *conv*$\varphi(x)$. The following statement gives a criterion of the equality of a polyhedral function to a convex envelope of a given function.

THEOREM 1.3. *Let function $f(x)$ be continuously differentiable on convex compact polytope P and its convex envelope conv$_P f(x)$ be a polyhedral function. Let there exist a collection of m affine function $h_i(x)$ so that each function h_i fills the conditions of Lemma 1.1. Then function conv$_P f(x)$ coincides with $\psi(x) = \max\{h_i(x)|i = 1,\ldots,m; x \in P\}$, $\psi(x) = \infty\,\forall x \notin P$ if and only if: α) The generating set of the function $\psi(x)$ coincides with vertP. β) For each vertex $\xi \in$ vertP there exists $i \in \{1,\ldots,m\}$ such that $h_i(\xi) = f(\xi)$.*

Proof. The necessary part of the theorem is self-evident. To prove the sufficiency, note that it follows from inequality (14) and the condition 'β' that $\psi(\xi) = f(\xi)\,\forall \xi \in$ *vertP*. From condition 'α' of the theorem and equality (10) one can conclude that *vert*(*epi*$(\psi)) =$ *vert*(*epi*(*conv*$_P(f)$)). Because any closed convex set is completely defined by its extreme points, $\psi(x) = conv_P f(x)\,\forall x \in P$. □

Now we will use Theorem 1.3 to prove (6). Let us consider the following function on u^n : $\varphi_m(x) = \sum_{1\leq j1<\cdots<jm\leq n} x_{j1}\ldots x_{jm}$ where $1 < m < n$. We will prove that:

$$conv\varphi_m(x) = \max\left\{0,{}_{k-1}C_{m-1}\sum_1^n x_i - (m-1)_kC_m|k = m,\ldots,n\right\}.$$

Note, that this formula coincides with (6) if $m = 2$, because $\varphi_2(x) = \varphi(x)$ by definition. We will check the condition of Lemma 1.1 for the elements of *conv*φ_m: $h_{m1}(x) = 0$; $h_k(x) ={}_{k-1} C_{m-1} \sum_1^n x_i - (m-1)_kC_m$ where $k = m,\ldots,n$. Let $x \in$ *vert*U^n and $n_x = \sum x_i$. Because $h_k(x) = n_x \bullet_{k-1} C_{m-1} - (m-1)_kC_m$ and $\varphi_m(x) = {}_{n_x}C_m$, $h_k(x) < \varphi(x)$ if $n_x < k-1$ or $n_x > k+1$. If $n_x = k$ or $n_x = k-1$, $h_k(x) = \varphi_m(x)$ and it is easy to check that there are $n+1$ linearly independent points among these vertices. It follows also from this consideration that condition 'β' is fulfilled. To check condition 'α' we prove that any system of equations: $h_k(x) = h_l(x), x_i = 1, x_j = 0; k,l,i,j \in 1,\ldots,n$ may have only solutions made of 0's and 1's. After collecting similar terms we will have only one equation $\sum x_i = n_x$ and all the other $n_x - 1$ equations will be of the form $x_i = 1$ or $x_j = 0$. □

Corollary 1.1 gives a general criterion of when the standard approach leads to the precise result. But, sometimes it is difficult to check if the condition (12) is fulfilled. The following Theorem 1.4 gives a condition sufficient to establish that the convex envelope of the sum of functions is equal to the sum of the convex envelopes of the addends.

THEOREM 1.4. *Let P be a Cartesian product of polytopes, $P = P_0 \times P_1 \times \dots P_k$, $P_i \in R^{n_i}$, and let $f_i(x_0, x_i)$ be a continuous function defined on $P_0 \times P_i$ $i = 1, ...k$. If each $f_i(x_0, x_i)$ is a concave function of x_0 when x_i is fixed and P_0 is a simplex*[⋆]*, then*

$$conv_P \left(\sum_1^m f_i(x_0, x_i) \right) = \sum_1^m conv\, f_i(x_0, x_i)\,. \tag{16}$$

Proof. Let $x_0 \in P_0$ and $x_{0\nu}$ enumerates the vertices of P_0, $\nu = 1, \dots, n_0 + 1$. There exist nonnegative values $\alpha_\nu \geq 0 : \sum_\nu \alpha_\nu x_{0\nu} = x_0; \sum_\nu \alpha_\nu = 1$ and vector $\alpha = \{\alpha_\nu\}$ is uniquely determined by the vertices $\{x_{0\nu}\}$. Using concavity of $f_i(x_0, x_i)$ of x_0, it is easy to prove that:

$$conv f_i(x_0, x_i) = \sum_0^{n_0+1} \alpha_\nu conv f_i(x_{0\nu}, x_i)\,. \tag{17}$$

On the other hand, $\sum_1^m f_i(x_0, x_i)$ is a convex function of x_0 as well, and as a result:

$$conv_P \left(\sum_1^m f_i(x_0, x_i) \right) = \sum_0^{n_0+1} \alpha_\nu \sum_1^m conv\, f_i(x_{0\nu}, x_i)\,. \tag{18}$$

Because $f_i(x_{0\nu}, x_i)$ are separable functions, $conv(\sum_1^m f_i(x_{0\nu}, x_i)) = \sum_1^m$ $conv\, f_i(x_{0\nu}, x_i)$ [10]. Thus, summation (17) over i leads to the equality (18). □

REMARK 1.6. Theorem 1.4 generalizes a well known results [10] giving a condition sufficient to ensure that the convex envelope of a set of functions equals to the sum of their convex envelopes in the separable case. In the separable case function $f_i(x_0, x_i)$ depend only on x_i and not on x_0. Note, that the requirement that P_0 be a simplex is essential in this proof, and the conclusion does not follow if $|vert\, P_0| > n_0 + 1$. In slightly different form this theorem was first presented in [14].

[⋆] I.e. the number of vertices of the polytope P_0 is $n_0 + 1$.

REMARK 1.7. It follows from Theorem 1.2 that if each function $f_i(x_0, x_i)$ is a component-wise concave function of x_i and x_0, then both *conv* $f_i(x_0, x_i)$ and *conv*$(\sum_1^m f_i(x_0, x_i))$ are polyhedral functions. Thus, combining Theorems 1.2 and 1.4 gives the criterion sufficient to ensure (16) for functions with polyhedral convex envelopes. In particular, let us consider the case of multilinear functions on U^n. It follows from Theorem 1.4 that if all the addends depend on the 1-dimensional common variable x_0, (for example, if $\sum f(x_0, x_i) = x_0x_1x_2 - x_0x_4x_5 + x_6x_7$) then the standard approach (4) leads to the convex envelope.

2. Convex Envelope Formula for Multilinear Functions

As was mentioned above, a lot of attention was devoted to linearization of the multilinear functions $f(x_1, \ldots, x_n)$ (1) on the unit hypercube U^n. Function (1) may be defined as an affine function of each of its scalar arguments $x_i \in U^1 = [0, 1]$ while all its other arguments are fixed, and $U^n = U^1 \times \cdots \times U^1$.

It seems reasonable to analyze the convex envelope properties for a straightforward generalization of the multilinear function (1) when we have a function $L(x_1, \ldots, x_n)$, which linearly depends on each of its *vector* arguments $x_1, \ldots, x_n$.

DEFINITION 2.1. Function $L(x_1, \ldots, x_k)$ is said to be a general multilinear function if for each $i = 1, \ldots, k$ function $L(x_1^0, \ldots, x_i, \ldots, x_k^0)$ linearly depends on vector x_i provided that all the other $k-1$ vector arguments are fixed*.

We will consider general multilinear functions $L(x), x = \{x_1, \ldots, x_k\}$ defined on the Cartesian product of convex polytopes: $x \in P = P_1 \times \cdots \times P_k, x_i \in P_i \subset R^{n_i}, i = 1, \ldots, k$. As it was stated in the remark 1.3 above, in this case *conv*$_P L(x)$ is a polyhedral function**.

DEFINITION 2.2. Let $L(x)$ be a general multilinear function defined on $R^{n_1} \times \cdots \times R^{n_k}$. For the function $L(x)$ and any given point $\xi = \{\xi_1, \ldots, \xi_k\} : \xi_i \in R^{n_i}, i = 1, \ldots, k$ the *associated affine function* $L_\xi(x)$ is defined by the following expression:

$$L_\xi(x) = \sum_{i=1,\ldots,k} L(\xi_1, \ldots, \xi_{i-1}, x_i, \xi_{i+1}, \ldots, \xi_k) - (k-1)L(\xi)\,. \qquad (19)$$

For example, for monomial function $L(x) = x_1^* \ldots^* x_k$ and $\xi = (1, \ldots, 1)$ function $L_\xi(x) = x_1^* 1^* \cdots^* 1 + \cdots + 1^* \cdots^* 1^* x_k - (k-1)1^* \cdots 1 = \sum_i x_i - (k-1)$. If $\nu = (0, \ldots, 0)$, function $L_\nu(x) = x_1^* 0 + \cdots + 0^* x_k - (k-1)^* 0 = 0$ (see (2) above).

* Thus, $L(x_1^0, \ldots, x_i, \ldots, x_n^0)$ is an affine function of x_i; for example function $L = x_1 + x_1x_2 + x_1x_2x_3$ is a general multilinear function.

** In the case of an arbitrary polytope P it is not so: if, for example, $L(x_1, x_2) = x_1x_2$ and $P = \{x | 0 \le x_1 \le x_2 \le 1, x_i \ge 0\}$, *conv*$_P L(x)$ is not polyhedral.

THEOREM 2.1. *Let $L(x_1,\ldots,x_k) : P \to R^n$ be a general multilinear function defined on the Cartesian product of convex polytopes: $P = P_1 \times \cdots \times P_k$, $x_i \in P_i \subset R^{n_i}$, $i = 1,\ldots,k$. Let $\xi = (\xi_1,\ldots,\xi_k)$ be a vertex of P: $\xi_i \in vertP_i$ and for the associated affine function (19) $L_\xi(x)$ fulfilled the inequality:*

$$L_\xi(x) \le L(x)\,\forall x \in \mathit{vert}P. \tag{20}$$

Then affine function $L_\xi(x)$ is an element of the conv$_P L(x)$.

Proof. We will show that both conditions (13) and (14) of Lemma 1.1 are fulfilled in this theorem. Condition (14) is equivalent to (20). To prove (13), let us define polytope S as a convex envelope of all points $x = (x_1,\ldots,x_k) \in P$, which have $k-1$ components equal to the corresponding components of the point $\xi = (\xi_1,\ldots,\xi_k)$, i.e.

$$\begin{aligned} S = \mathit{conv}\{&(\xi_1,\ldots,\xi_k);(x_1,\xi_2,\ldots,\xi_k);\ldots;\\ &(\xi_1,\ldots,\xi_{k-1},x_k)|x_i \in P_i, i = 1,\ldots,k\}\end{aligned}$$

The set S is an $\sum n_i$-dimensional polytope, and its vertices are: $(\xi_1,\ldots,\xi_k)$, $(\eta_1,\ldots,\xi_k),\ldots,(\xi_1,\ldots,\eta_k)$ where $\eta_i \in \mathit{vert}P_i$. To check the equality (13) it is sufficient to show that $L_\xi(x) = L(x)\,\forall x \in \mathit{vert}S$. If we calculate $L(x)$ for $x = (\xi_1,\ldots,\xi_{i-1},\eta_i,\xi_{i-1},\ldots,\xi_k)$ we will see after collecting the similar terms that $L_\xi(x) = (i-1)L(\xi) + L(x) + (k-i)L(\xi) - (k-1)L(\xi) = L(x)$. □

REMARK 2.1. It is easy to see that both convex envelope formulas (2) and (3) may be considered as particular cases of Theorem 2.1. For example, for the case of multilinear function $L(x) = -x_1^* \ldots^* x_k$ if $\xi = \{\xi|\xi_j = 1, j \ne i, j = 1,\ldots,k$ and $\xi_i = 0\}$ the associated affine function $L_\xi(x) = -x_i$ which is exactly the i-th element of the *conv*$_{U^n} L(x)$ (see (3)).

The first and the last elements of the *conv*$_{U^n} \sum_{i>j} x_i x_j$ defined with (6) may be also considered as associated affine functions $L_\xi(x)$ and $L_\eta(x)$ corresponding to the points $\xi = \{0,\ldots,0\}$ and $\eta = \{1,\ldots,1\}$.

REMARK 2.2. The question arises: Is it true that convex envelopes of general multilinear functions always contain elements which are associated affine functions (defined by (19)). The answer is negative, as it follows from the following example for the bilinear function $L(x) = x_1x_2 + x_2x_3 - x_1x_3$ on U^3. For each $\xi \in \mathit{vert}U^3$ there exists another vertex, ν such that $L_\xi(\nu) > L(\nu)$. For example, if $\xi = (1,0,1), L_\xi(x) = -x_1 + 2x_2 - x_3 + 1 > L(x)$ at $x = (0,1,0)$.

In the case of the bilinear functions, when $L(x) = x_1Ax^2$, $P = P_1 \times P_2$, $x_1 \in P_1, x^2 \in P_2$ if $\xi = \{\eta,\nu\}$: $\eta \in \mathit{vert}P_1, \nu \in \mathit{vert}P_2$, the associated affine function $L_\xi(x) = \eta Ax^2 + x_1A\nu - \eta A\nu$. In this case, the condition (20) which is necessary and sufficient for the function $L_\xi(x)$ to be an element of the *conv*$_P L(x)$ may be rewritten in the equivalent form:

$$(x_1 - \eta)A(x^2 - \nu) \ge 0 \quad \forall x_1 \in P_1, \quad x^2 \in P_2 \tag{21}$$

For example, if (1) is a bilinear function and set $N_- = \emptyset$, then both the 'minimal' ($\xi = \{0, \ldots, 0\}$) and the 'maximal' ($\eta =$"$\{1, \ldots, 1\}$) vertices generate the associated affine functions $L_\xi(x)$ and $L_\eta(x)$ which are elements of the $conv_P L(x)$.

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References

1. F.A. Al-Khayyal and J.E. Falk, Jointly constrained biconvex programming. *Mathematics of Operations Research* **8** (1983), 272–286.
2. E. Balas and J.B. Mazzola, Nonlinear 0–1 programming: I. Linearization techniques, *Mathematical Programming* **30** (1984), 1–21.
3. E. Balas and J.B. Mazzola, Nonlinear 0–1 programming: II. Dominance relations and algorithms, *Mathematical Programming* **30** (1984), 22–45.
4. Y. Crama, Concave extension for nonlinear 0–1 maximization problems, *Mathematical Programming* **61** (1993), 53–60.
5. Y. Crama, Recognition problems for polynomials in 0–1 variables, *Mathematical Programming* **44** (1989), 139–155.
6. F. Glover and E. Woolsey, Converting the 0–1 polynomial programming problem to a linear 0–1 program, *Operations Research* **22** (1974), 180–182.
7. P.L. Hammer, P. Hansen and B. Simeone, Roof duality, complementation and persistency in quadratic 0–1 optimization, *Mathematical Programming* **28** (1984), 121–155.
8. P.L. Hammer and S. Rudeanu, *Boolean Methods in Operations Research and Related Areas* (Springer, Berlin, New York, 1968).
9. P. Hansen and B. Simeone, On the equivalence of paved duality and standard linearization in nonlinear 0–1 optimization, *Discrete Applied Mathematics* **29** (1990), 187–193.
10. R. Horst and H. Tuy, *Global Optimization, Deterministic Approaches* (Springer-Verlag, Berlin, 1993).
11. A. Ioffe and V. Tihomirov, *Teoria extremal'nih zadach* (Theory of optimization problems), Moscow, Nauka (Science), 1974 (in Russian).
12. M. Minoux *Programmation mathematique. Theorie et algorithmes dunod*, Bordas et C.N.E.T.-E.N.S.T., Paris, 1989.
13. P. Pardalos, On the passage from local to global optimization, In: J.R. Birge and K.G. Murty, *Mathematical Programming: The State of the Art* (The University of Michigan, 1994, p. 220–247).
14. A.D. Rikun, *Branch-and-Bound Method in Solution of Some Non-Convex Mathematical Programming Problems in Water Pollution Control.* Ph.D. Thesis, USSR Academy of Sci. Computer Center, Moscow, 1982.
15. R.T. Rockafellar, *Convex Analysis* (Princeton University Press, Princeton, NJ, 1970).
16. H.D. Sherali and A. Alameddine, An Explicit Characterization of the Convex Envelope of a Bivariate Bilinear Function over Special Polytopes, *Annals of Operations Research*, Computational Methods in Global Optimization, Eds. P.M. Pardalos and J.B. Rosen, Vol. 25, pp. 197–210, 1990.
17. H.D. Sherali and C.H. Tuncbilek, A Global Optimization Algorithm for Polynomial Programming Problems Using a Reformulation-Linearization Technique, *Journal of Global Optimization* **2** (1992), 101–112.